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THE DISTRIBUTION OF SUMS OF DEPENDENT LOG-NORMAL VARIABLES.(U)
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THE DISTRIBUTION OF SUMS OF DEPENDENT LOG-NORMAL VARIABLES

by

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I. Introduction

Distributions of sums of dependent or independent log-normal random variables appear in various fields of applied statistics. Concentrations of pollutants in the air or in water, life distributions of components in reliability systems and other applications. The available expression for the distributions -f sums of log-normal variables are complicated and in-attractive. Simpler expressions are needed for practical applications. The purpose of the present paper is to provide some simple approximations and algorithms that can be easily applied for obtaining numerical results. We are concerned with two types of variables (i) $W = e^{X_1} + e^{X_2}$ and (ii) $Z = \log(e^{X_1} + e^{X_2})$, where X_1 and X_2 have a bivariate normal distribution. The dependence of the log-normal variables e^{X_1} and e^{X_2} is a function of the correlation ρ between X_1 and X_2. Naus [5] derived the moment generating function of Z for the case of $\rho = 0$ and equal variances of X_1 and X_2. Hamdan [2] extended Naus results to the case of arbitrary ρ and

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unequal variances. Lowrimore and Tsokos [4] derived the probability density function of Z and of W. Further elaboration of these distributions is given in the paper of Tsokos [6]. The problem is however, that the analytic expressions derived by Lowrimore and Tsokos are very complicated and require also numerical integration. Our aim is to develop simpler methods. In Section 2 we develop two numerical procedures for the determination of the distribution of W with arbitrary parameters. We present some numerical computations of the c.d.f. of W. These computations were performed according to a FORTRAN program given in Appendix A. In Section 3 we study the moments of W and in Section 4 we derive an approximation to the distribution of W based on the lognormal distribution having the same mean and variance. As demonstrated by numerical examples this approximation is very effective when the correlation between X_1 and X_2 is nonnegative. The lognormal approximation to the distribution of W, which is the normal approximation to the distribution of Z, does not provide very good results in the range of correlations close to -1. We tried therefore to correct for the pronounced skewness in the distribution of $Z = \log W$, when ρ is close to -1, by employing the Edgeworth expansion. For this purpose we have to determine the moments of $Z = \log W$. In Section 5 we discuss the problem of determining the moments of Z, in the case of X_1 and X_2 having a standard bivariate normal distribution. For the first moment an analytic expression similar to that of Hamdan [2] is given. We present however, an expression which is more suitable for numerical computations. For higher moments we provide a formula for a numerical approximation. The goodness of this approximation is also studied. Numerical computations show that the Edgeworth type of expansion mentioned earlier does not provide in the standard case any substantial improvement.

2. The Distribution of W.

Let X_1 and X_2 be random variables having a bivariate normal distribution with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, respectively and coefficient of correlation ρ. $-\infty < \mu_1, \mu_2 < \infty$; $0 < \sigma_1, \sigma_2 < \infty$ and $-1 \leq \rho \leq 1$. Let $W = e^{X_1} + e^{X_2}$ and let $F_{\underline{\theta}}(w)$ be the c.d.f. of W, under $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. Obviously, $F_\theta(w) = 0$ for all $w \leq 0$. For $w > 0$ we have for all $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$

$$F_{\underline{\theta}}(w) = P_{\underline{\theta}}[e^{X_1} + e^{X_2} \leq w] \tag{2.1}$$

$$= \frac{1}{\sigma_1} \int_{-\infty}^{\log w} P[e^{X_2} \leq w - e^y | X_1 = y] \; \phi(\frac{y-\mu_1}{\sigma_1})dy$$

where $\phi(\mu)$ denotes the standard normal p.d.f. The conditional distribution of X_2, given $X_1 = y$, is the normal distribution with mean $E\{X_2|X_1 = y\} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(y - \mu_1)$ and variance $V\{X_2|X_1 = y\} = \sigma_2^2(1 - \rho^2)$. Thus, if $-1 < \rho < 1$,

$$F_{\underline{\theta}}(w) = \frac{1}{\sigma_1} \int_{-\infty}^{\log w} \Phi(\frac{\log(w-e^y) - \mu_2 - \rho\frac{\sigma_2}{\sigma_1}(y-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}). \tag{2.2}$$

$$\phi(\frac{y - \mu_1}{\sigma_1})\; dy,$$

where $\Phi(u)$ is the standard normal c.d.f. The following are some special cases:

(i) If $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$ then the expressions are slightly simplified since e^μ is a scale parameter of the distribution and

$$F_{(\mu,\sigma,\mu,\sigma,\rho)}(w) = F_{(0,\sigma,0,\sigma,\rho)}(\frac{w}{e^\mu}),\; 0 \leq w \leq \infty.$$

(ii) If $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\rho = 1$ then W has a lognormal distribution. Indeed, in this case $X_2 = X_1$ with probability 1 and

$$(2.3) \qquad P[W \leq w] = P[e^{X_1} \leq \frac{w}{2}] = \Phi(\frac{\log w - \mu'}{\sigma}),$$

where $\mu' = \mu_1 + \log 2$.

(iii) When $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\rho = -1$ then $X_2 = -X_1$ with probability 1 and the distribution of W is given by,

$$(2.4) \qquad F_{\underline{\theta}}^{(-1)}(w) = \begin{cases} 0 & \text{, if } w \leq 2, \\ \Phi(\frac{\xi_2(w)-\mu_1}{\sigma_1}) - \Phi(\frac{\xi_1(w)-\mu_1}{\sigma_1}) & \text{, if } w > 2, \end{cases}$$

where $\xi_1(w) = \frac{1}{2}(w - \sqrt{w^2-4})$ and $\xi_2(w) = \frac{1}{2}(w + \sqrt{w^2-4})$. Notice that $e^x + e^{-x} \geq 2$ for all x.

2.1 Numerical Determination of the Distribution of W.

The integrand of (2.2) can be easily computed for each y value. A numerical integration of (2.2) over the range $(\mu_1 - 4.5\ \sigma_1, \log w)$ can then be readily executed. Notice that $\Phi(-4.5) = .34 \times 10^{-5}$. Therefore, the error committed by neglecting the range of $y < \mu_1 - 4.5\ \sigma_1$ is smaller than $.34 \times 10^{-5}$. For this reason, for values of w smaller than $e^{\mu_1 - 4.5\ \sigma_1}$, we approximate the value of $F_{\underline{\theta}}(w)$ by 0.

An m-point approximation to (2.2) is given by

$$(2.5) \qquad F_{\underline{\theta}}(w) = \sum_{j=1}^{m} \Phi\left(\frac{\log(w-e^{\eta'_j}) - (\mu_2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1) - \rho\frac{\sigma_2}{\sigma_1}\eta'_j}{\sigma_2\sqrt{1-\delta^2}}\right).$$

$$\left[\,\Phi\left(\frac{\eta_j - \mu_1}{\sigma_1}\right) - \Phi\left(\frac{\eta_{j-1} - \mu_1}{\sigma_1}\right)\right],$$

where

$$(2.6) \qquad \eta_j = \mu_1 - 4.5\,\sigma_1 + j\,\Delta(w)\,, \qquad j = 1,\ldots,m$$

$$\Delta(w) = (\log w - \mu_1 + 4.5\,\sigma_1)/m$$

and

$$\eta'_j = \eta_j - \Delta(w)/2.$$

In Tables 1 we provide the results of computing $F_\theta(w)$ according to (2.5) with $m = 20$ and subintervals for the case of $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\rho = -.99(.33).99$.

Theoretically, the limit of (2.5) as $m \to \infty$ is the integral (2.2). We see in Table 1 that the differences between the results of computations with $m = 20$ and $m = 80$ are in most cases in the third decimal place. It is not difficult, however, to determine the distribution functions very accurately by applying the method with a large m value. The computation of the seven distributions of Table 1, with $m = 80$ required about 60 seconds (double precision) on a relatively slow computer (Honeywell GE-400). A Fortran program according to which these computations were performed is available upon request. To evaluate the goodness of the approximation with $m = 80$ we provide in Table 2 a comparison of the results obtained for the case of $\rho = 1$ from (2.5) against the exact log-normal distribution, given by formula (2.3).

Table 1. Distribution of W for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$. Determined according to (2.5) for m = 80 and m = 20.

m	W/ρ	-.99	-.66	-.33	0	-.33	.66	.99
	1.000	0.00000	0.02006	0.06698	0.11346	0.15739	0.19994	0.24281
	2.000	0.12205	0.29188	0.35051	0.39414	0.48153	0.46590	0.49900
	3.000	0.66113	0.59716	0.59660	0.60781	0.62269	0.63926	0.65685
	4.000	0.81100	0.77229	0.75013	0.74334	0.74393	0.74856	0.75560
	5.000	0.88036	0.86294	0.84076	0.82774	0.82141	0.81938	0.82013
	6.000	0.92414	0.91197	0.89489	0.88122	0.87221	0.86683	0.86402
80	7.000	0.94697	0.94042	0.92823	0.91600	0.90645	0.89961	0.89490
	8.000	0.96162	0.95799	0.94947	0.93922	0.93014	0.92287	0.91726
	9.000	0.97150	0.96940	0.96345	0.95512	0.94690	0.93976	0.93382
	10.000	0.97838	0.97712	0.97293	0.95901	0.95901	0.95228	0.94635
	1.000	0.00000	0.02004	0.06696	0.11353	0.15755	0.20028	0.24416
	2.000	0.12185	0.29124	0.35052	0.39417	0.43135	0.46579	0.49262
	3.000	0.66129	0.59864	0.59759	0.60813	0.62229	0.63864	0.65854
	4.000	0.82205	0.77607	0.75199	0.74399	0.74351	0.74765	0.76265
	5.000	0.88671	0.86693	0.84293	0.82859	0.82105	0.81835	0.82802
	6.000	0.92364	0.91506	0.89697	0.88215	0.87193	0.86578	0.87066
20	7.000	0.94654	0.94247	0.93004	0.91692	0.90625	0.89860	0.89974
	8.000	0.96123	0.95924	0.95095	0.94008	0.93000	0.92193	0.92038
	9.000	0.97115	0.97012	0.96463	0.95589	0.94681	0.93890	0.93555
	10.000	0.97804	0.97751	0.97385	0.96693	0.95897	0.95150	0.94701

Table 2. The Distribution of W for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = 1$.

W	(2.3)	(2.5)
1.000	0.2441	0.2498
2.000	0.5000	0.4916
3.000	0.6574	0.6550
4.000	0.7559	0.7655
5.000	0.8202	0.8218
6.000	0.8640	0.8608
7.000	0.8949	0.8889
8.000	0.9172	0.9224
9.000	0.9337	0.9367
10.000	0.9462	0.9476

2.2 Gauss-Legendre Quadrature.

The numerical integration method prescribed in Section 2.1 is quite simple in the sense that it is based on subintervals of equal length. The results obtained seems to be quite stable over all the range of $-1 \leq \rho \leq 1$. However, as seen in Table 2, there is some difference although small, between the numerical results obtained and what should be obtained in the case of $\rho = 1$. We therefore investigate here what numerical results can be obtained by applying the Gauss-Legendre quadrature formula, with 80 cut-points, for integrating (2.2) numerically. An m-point Gauss-Legendre quadrature formula is

$$(2.7) \qquad \int_a^b f(x)dx = (b-a) \sum_{i=1}^{m} p_i \, f(x_i) + R_m,$$

where $x_i = \frac{b-a}{2} \xi_i + \frac{a+b}{2}$, $i=1,\ldots,m$ ξ_i is the i-th zero of the Legendre polynomial $P_m(\xi)$ over $-1 \leq \xi \leq 1$, and $p_i = 1/(1-\xi_i^2)[P_m'(\xi_i)]^2$ is a weight assigned to $\xi_i (i=1,\ldots,m)$. R_m is a proper remainder term (see Abramowitz and Segun [8; pp. 888]). The values of ξ_i and $2p_i$ for $m = 80$ are tabulated in Abramowitz and Segun [8; pp. 918]. For the case under consideration, let

$$(2.8) \qquad f(x;\alpha,\beta,s,w) = \frac{1}{s}\left[\log(w-e^x) - \alpha - \beta x\right], \; -\infty \leq x \leq \log w;$$

where $\beta = \rho\sigma_2/\sigma_1$, $\alpha = \mu_2 - \beta\mu_1$ and $s^2 = \sigma_2^2(1-\rho^2)$. By simple change of variables, we can write the c.d.f. of W in the form

$$(2.9) \qquad F_{\underline{\theta}}(w) = \int_0^{\Phi\left(\frac{\log w - \mu_1}{\sigma_1}\right)} \Phi(f(\mu_1 + \sigma_1 \Phi^{-1}(y); \alpha,\beta,s,w))dy.$$

Hence, the Gauss-Legendre approximation is, according to (2.7)

$$(2.10)\qquad F_{\underline{\theta}}(w) \cong \Phi\left(\frac{\log w - \mu_1}{\sigma_1}\right) \sum_{i=1}^{m} p_i\, \Phi(f(\mu_1 + \sigma_1\, \Phi^{-1}(y_i);\ \alpha,\beta,s,w))$$

where $y_i = \frac{1}{2}\,\Phi\left(\frac{\log w - \mu_1}{\sigma_1}\right)(1 + \xi_i)$, $i=1,\ldots,m$.

In Table 3 we present the results of the numerical determination of the distributions corresponding to those of Table 1, according to the Gauss-Legendre method with $m = 80$.

The comparisons of Table 1 and 3 show that the two methods yield very close results. In Table 4 we provide further comparisons of the two methods in non-standard cases.

Further simplication of the calculations without sacrificing much accuracy can be achieved by applying formula (2.10) for a small value of m. We have seen in Table 1 that (2.5) provides highly accurate results with $m = 20$. For small values of m formula (2.5) may not yield sufficiently accurate results, as shown in Table 5, since it is based on a partition to equal size subintervals.

On the other hand, formula (2.10) with $m = 6$ yields accurate results when $|\rho|$ is not too close to 1. This is seen in Table 6. For $\rho = 1$ and $\rho = -1$ we can compute the distributions exactly by other formulae.

For $m = 6$ formula (2.6) should be used with the following constants (see Abramowitz and Segun [8, pp. 921]).

i	$\frac{1 + \xi_i}{2}$	p_i
1	.03376	.08566
2	.16939	.18038
3	.38069	.23395
4	.61930	.23395
5	.83060	.18038
6	.96623	.08566

Table 3. The Distribution of W for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$. Determined According to The Gauss-Legendre Quadrature; m = 80.

W/ρ	-.99	-.66	-.33	0	.33	.66	.99
1.000	0.00000	0.02006	0.06698	0.11345	0.15737	0.19992	0.24278
2.000	0.12207	0.29191	0.35054	0.39416	0.43153	0.46590	0.49900
3.000	0.66176	0.59728	0.59668	0.60785	0.62272	0.63928	0.65689
4.000	0.81144	0.77245	0.75022	0.74340	0.74398	0.74860	0.75565
5.000	0.88254	0.86301	0.84081	0.82780	0.82147	0.81943	0.82019
6.000	0.92191	0.91186	0.89489	0.88127	0.87227	0.86689	0.86409
7.000	0.94567	0.94010	0.92817	0.91603	0.90651	0.89967	0.89497
8.000	0.96088	0.95759	0.94935	0.93923	0.93019	0.92292	0.91732
9.000	0.97105	0.96910	0.96330	0.95510	0.94695	0.93981	0.93388
10.000	0.97810	0.97699	0.97281	0.96621	0.95905	0.95233	0.94641

We remark that for m = 6 formula (2.10) can be used also with hand calculators and tables of the standard normal distributions.

Table 4. The Distribution of W Computed According to (2.5) and According to Gauss-Legendre Quadrature.

Case I: $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1 = 1$, $\sigma_2 = 3$; Case II: $\mu_1 = 0$, $\mu_2 = 3$, $\sigma_1 = 1$, $\sigma_2 = 4$.

Case I	W/ρ	-.99	-.66	-.33	0	.33	.66	.99
	1.000	0.00000	0.06995	0.13479	0.18889	0.23925	0.29053	0.34915
	3.222	0.49500	0.47368	0.49128	0.51462	0.53951	0.56491	0.58869
	5.444	0.65530	0.64135	0.63729	0.64234	0.65173	0.66301	0.67386
	7.667	0.72296	0.71551	0.70896	0.70810	0.71119	0.71640	0.72209
	9.889	0.76153	0.75694	0.75124	0.74855	0.74890	0.75123	0.75444
	12.111	0.78723	0.78404	0.77948	0.77636	0.77546	0.77630	0.77815
G-L	14.333	0.80602	0.80363	0.79999	0.79693	0.79547	0.79548	0.79653
	16.556	0.82063	0.81874	0.81579	0.81295	0.81125	0.81079	0.81134
	18.778	0.83245	0.83091	0.82847	0.82589	0.82413	0.82339	0.82360
	21.000	0.84232	0.84103	0.83897	0.83665	0.83489	0.83398	0.83390
	1.000	0.00000	0.07007	0.13503	0.18916	0.23950	0.29067	0.34916
	3.223	0.49724	0.47431	0.49169	0.51490	0.53965	0.56490	0.58866
	5.444	0.65536	0.64165	0.63751	0.64249	0.65179	0.66297	0.67380
	7.667	0.72293	0.71559	0.70907	0.70819	0.71122	0.71638	0.72204
	9.889	0.76147	0.75693	0.75126	0.74858	0.74891	0.75119	0.75437
(2.5)	12.111	0.78715	0.78400	0.77945	0.77635	0.77545	0.77625	0.77807
	14.333	0.80594	0.80359	0.79994	0.79690	0.79545	0.79544	0.79645
	16.556	0.82055	0.81870	0.81574	0.81291	0.81123	0.81075	0.81125
	18.778	0.83237	0.83087	0.82842	0.82585	0.82409	0.82335	0.82352
	21.000	0.84223	0.84099	0.83892	0.83660	0.83485	0.83395	0.83390
Case II								
	0.135	0.00000	0.00027	0.00367	0.00992	0.01658	0.02143	0.02274
	0.368	0.00000	0.02169	0.05715	0.08922	0.11791	0.14273	0.15794
	1.000	0.15783	0.24605	0.29959	0.34449	0.38681	0.43029	0.48110
	2.781	0.67301	0.65902	0.66060	0.67334	0.69236	0.71697	0.74987
G-L	7.839	0.86972	0.86739	0.86341	0.86091	0.86120	0.86402	0.86784
	20.086	0.93147	0.93104	0.93015	0.92878	0.92735	0.92612	0.92522
	54.598	0.95984	0.95977	0.95957	0.95922	0.95875	0.95816	0.95757
	148.413	0.97724	0.97725	0.97719	0.97710	0.97698	0.97682	0.97657
	403.429	0.98777	0.98781	0.98777	0.98774	0.98772	0.98770	0.98768
	0.135	0.00000	0.00027	0.00367	0.00995	0.01662	0.02149	0.02278
	0.368	0.00000	0.02176	0.05734	0.08949	0.11825	0.14321	0.15865
	1.000	0.15748	0.24682	0.30027	0.34516	0.38752	0.43113	0.48035
	2.781	0.68028	0.66717	0.66809	0.68012	0.69846	0.72228	0.75381
(2.5)	7.389	0.86966	0.86737	0.86346	0.86100	0.86128	0.86403	0.86777
	20.086	0.93140	0.93099	0.93012	0.92876	0.92730	0.92607	0.92515
	54.598	0.95977	0.95971	0.95954	0.95921	0.95870	0.95810	0.95749
	148.413	0.97717	0.97719	0.97717	0.97709	0.97695	0.97675	0.97650
	403.429	0.98772	0.98775	0.98775	0.98774	0.98770	0.98763	0.98753

Table 5. The Distribution of W for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$. According to (2.5) with m = 6.

W/ρ	-.99	-.66	-.33	0	.33	.66	.99
1.000	0.00000	0.02050	0.07004	0.11845	0.16275	0.20154	0.22662
2.000	0.09668	0.30755	0.36506	0.40738	0.44136	0.46498	0.43157
3.000	0.65238	0.61829	0.61385	0.62212	0.63209	0.63592	0.56577
4.000	0.84762	0.78664	0.76445	0.75557	0.75147	0.74380	0.65754
5.000	0.87245	0.87047	0.85124	0.83737	0.82709	0.81399	0.72333
6.000	0.89576	0.91525	0.90219	0.88858	0.87638	0.86130	0.77234
7.000	0.94787	0.94138	0.93322	0.92157	0.90948	0.89421	0.80998
8.000	0.96941	0.95779	0.95285	0.94344	0.93232	0.91775	0.83961
9.000	0.97429	0.96867	0.96574	0.95832	0.94847	0.93500	0.86339
10.000	0.97656	0.87618	0.97449	0.96871	0.96014	0.94789	0.88276

Table 6. The Distribution of W Determined by Formula (2.10).
Case I: $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1 = 1$, $\sigma_2 = 1$; Case II: $\mu_1 = 0$, $\mu_2 = 3$, $\sigma_1 = 1$, $\sigma_2 = 4$; m = 6.

Case I	W/ρ	0.99	-.66	-.33	0	.33	.66	.99
	1.000	0.00000	0.02007	0.06714	0.11344	0.15720	0.19994	0.24896
	2.000	0.11675	0.29190	0.35060	0.39388	0.43139	0.46605	0.53030
	3.000	0.63077	0.59662	0.59623	0.60685	0.62180	0.63946	0.63481
	4.000	0.83510	0.77805	0.75230	0.74339	0.74276	0.74764	0.74155
	5.000	0.86792	0.87131	0.84572	0.83008	0.82143	0.81729	0.85772
	6.000	0.90840	0.91886	0.90063	0.88528	0.87397	0.86476	0.88071
	7.000	0.95439	0.94530	0.93345	0.92085	0.90982	0.89886	0.89062
	8.000	0.97747	0.96143	0.95368	0.94406	0.93451	0.92393	0.89707
	9.000	0.98548	0.97190	0.96666	0.95854	0.95177	0.94264	0.90196
	10.000	0.98922	0.97902	0.97531	0.97008	0.96394	0.95656	0.91261
Case II								
	0.135	0.00000	0.00027	0.00369	0.00995	0.01662	0.02150	0.02275
	0.368	0.00000	0.02173	0.05731	0.08935	0.11806	0.14301	0.15864
	1.000	0.15551	0.24696	0.30052	0.34510	0.38741	0.43104	0.47818
	2.718	0.65624	0.66079	0.66231	0.67452	0.69346	0.71796	0.76278
	7.389	0.88892	0.86865	0.86445	0.86192	0.86259	0.86519	0.88895
	20.086	0.91415	0.93262	0.93062	0.92898	0.92835	0.92802	0.91327
	54.598	0.94280	0.96146	0.95990	0.95916	0.95924	0.96007	0.93661
	148.413	0.99226	0.97885	0.97742	0.97701	0.97725	0.97850	0.99119
	403.429	0.99982	0.98913	0.98791	0.98765	0.98787	0.98906	0.99982

3. The Moments of W and Some Characteristics of Its Distribution

As seen in the various tables of Section 2, the distribution of W is considerably skewed in non-standard cases. We develop here formulae for the moments of W and measures of skewness, kurtosis and other characteristics.

The r-th moment of W is given by,

$$(3.1)\qquad M_r(\underline{\theta}) = E_{\underline{\theta}}\{W^r\} = E_{\underline{\theta}}\{\sum_{j=0}^{r}\binom{r}{j}\exp\{(r-j)X_1 + jX_2\}\}$$

$$= \sum_{j=0}^{r}\binom{r}{j}\exp\{j\mu_2 + (r-j)\mu_1 + \frac{1}{2}((r-j)^2\sigma_1^{\,2} + 2\sigma_1\sigma_2\,\rho j(r-j) + \sigma_2^{\,2}\,j^2\,)\}.$$

Indeed, for each $j=0,\ldots,r$,

$$(r-j)X_1 + jX_2 \sim N((r-j)\mu_1 + j\mu_2,\ (r-j)\sigma_1^2 + 2j(r-j)\rho\sigma_1\,\sigma_2 + j^2\,\sigma_2^{\,2}).$$

Moreover, $E\{e^{N(\xi,\tau^2)}\} = \exp\{\xi + \tau^2/2\}$.

The central moments of W are denoted by $M_r^{\,*}(\underline{\theta})$ and are given in terms of $M_r(\underline{\theta})$ by the formula

$$(3.2)\qquad M_r^{\,*}(\theta) = \sum_{j=0}^{r}\binom{r}{j}(-1)^j\,M_{r-j}(\underline{\theta})(M_1(\underline{\theta}))^j.$$

When X_1 and X_2 have the same marginal distributions, i.e., $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$, then e^{μ} is a scale parameter of the distribution of W and we have

$$(3.3)\qquad \begin{aligned} M_1(\underline{\theta}) &= e^{\mu}\cdot 2\,e^{\sigma^2/2} \\ \text{s.d.}_W(\underline{\theta}) &= e^{\mu}\sqrt{2}\,e^{\sigma^2/2}\,(e^{\sigma^2} + e^{\rho\sigma^2} - 2)^{1/2}, \end{aligned}$$

where $\text{s.d.}_W(\underline{\theta})$ is the standard deviation of W, under $\underline{\theta}$.

In Figure 1 we illustrate the standard deviation of W, as a function of ρ, for $\sigma = 1$, 1.5 and 2, and $\mu = 0$, on a logarithmic scale.

We see in Figure 1 that the standard deviation of W changes very slowly when $\rho \leq 0$ and increases faster over the range of positive ρ values. Also, the relative rate of increase grows fast with σ. In other words, for $\sigma = 2$ $s.d._W(\underline{\theta})$ is relatively constant over $\rho \leq 0$, compared to the case with $\sigma = 1$.

Other parameters of interest are the coefficients of skewness, $\gamma_1(\underline{\theta}) = (M_3^*(\theta))^2/(M_3^*(\theta))^3$ and of kurtosis $\gamma_2(\theta) = M_4^*(\theta)/(M_2^*(\theta))^2$. Again, when $\mu_1 = \mu_2 = \mu$ these parameters do not depend on μ (or on the scale parameter e^{μ}). For $\sigma_1 = \sigma_2 = \sigma$ we obtain that

$$(3.4) \qquad M_3^*(\underline{\theta}) = 2e^{\frac{3}{2}\sigma^2}[e^{3\sigma^2} + 3e^{\sigma^2}(e^{2\sigma^2\rho} - 2) - 6e^{\sigma^2\rho} + 8]$$

and

$$(3.5) \qquad M_4^*(\underline{\theta}) = 2e^{2\sigma^2}[e^{2\sigma^2}(e^{4\sigma^2} + 4e^{\sigma^2+3\sigma^2\rho} + 3e^{4\sigma^2\rho}) - 8e^{\sigma^2}(e^{2\sigma^2} + 3e^{2\sigma^2\rho})$$
$$+ 24e^{\sigma^2}(e^{\sigma^2} + e^{\sigma^2\rho}) - 24].$$

The coefficients of skewness and kurtosis, $\gamma_1(\underline{\theta})$ and $\gamma_2(\theta)$ are plotted in Figure 2 as functions of ρ for cases of $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2 = 1$, 1.5, 2.0. These plots show that the distribution of W becomes extremely skewed and flat when σ becomes large.

Figure 1. The Standard Deviation of W for μ = 0. For σ = 1, 1.5, 2.0 .

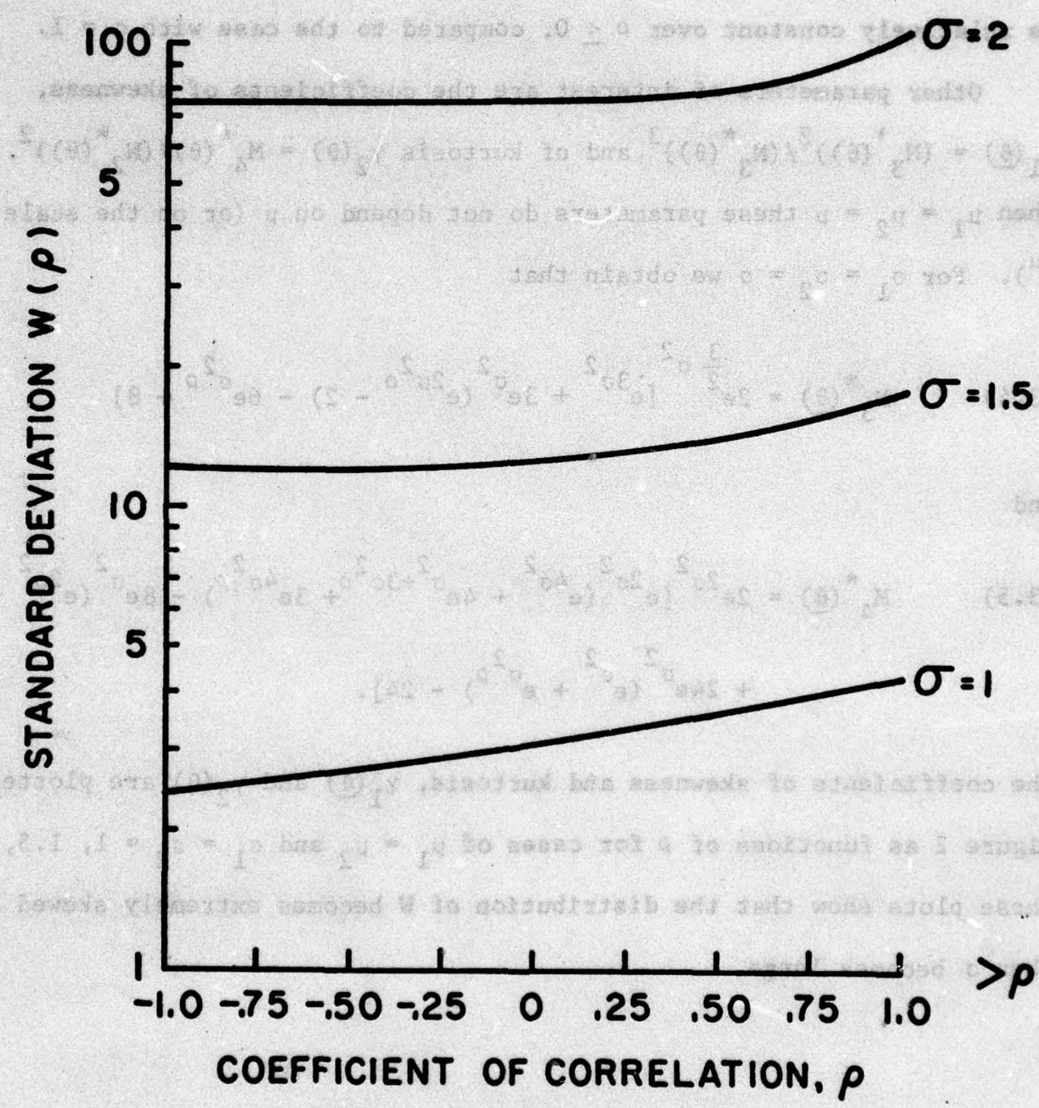


Figure 2. $\gamma_1(\underline{\theta})$ and $\gamma_2(\theta)$ for $\mu_1 = \mu_2$.

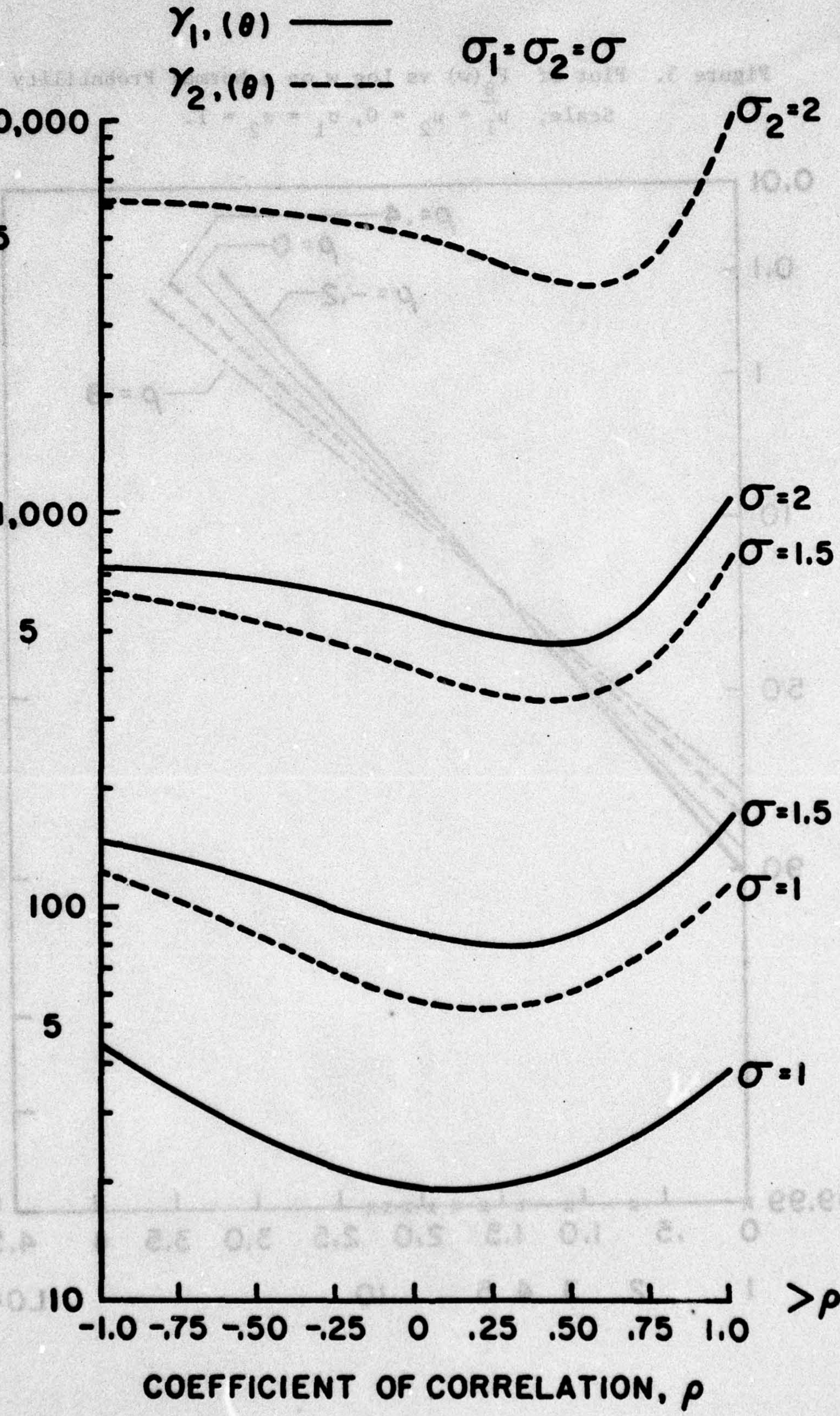
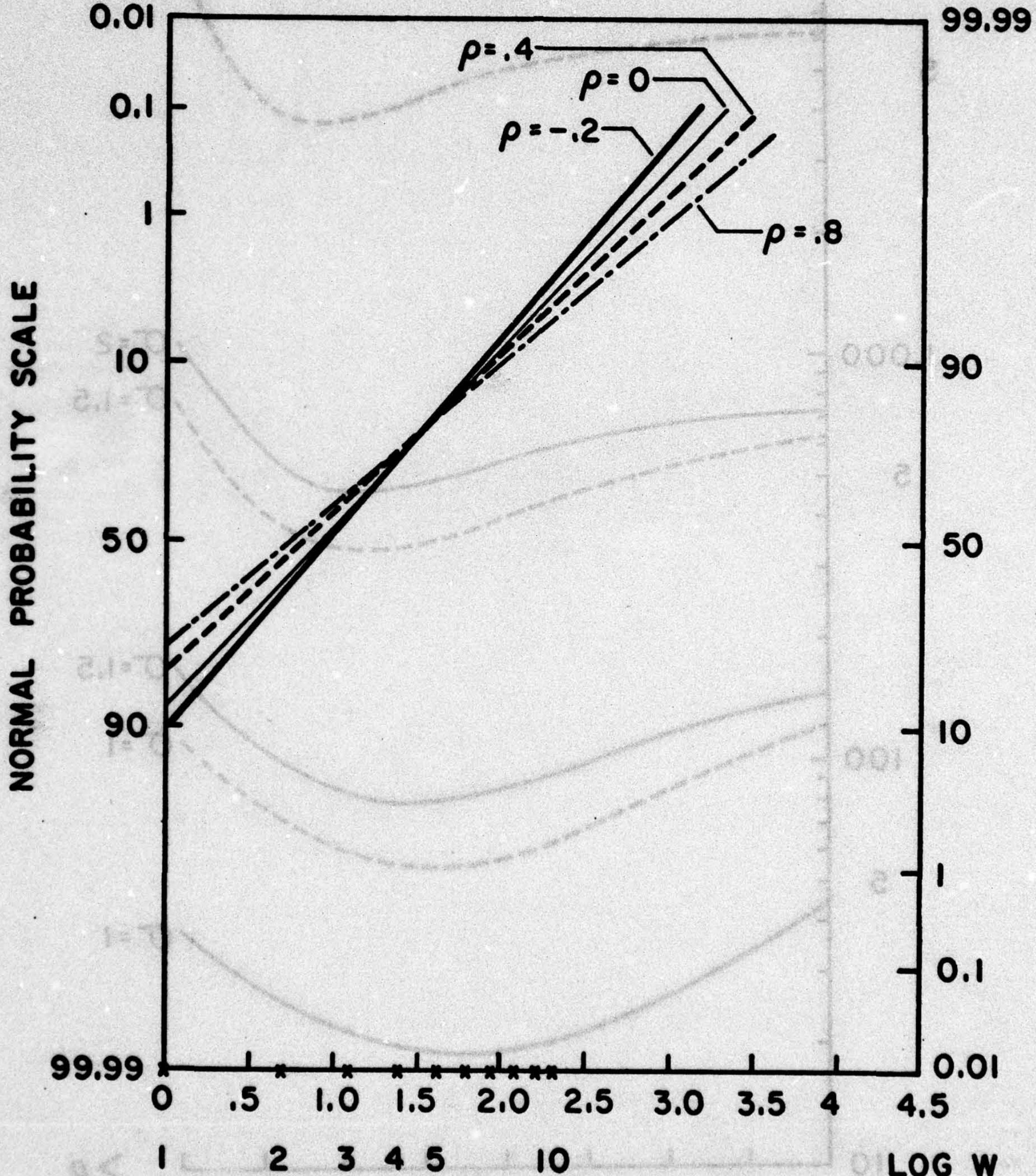


Figure 3. Plot of $F_{\theta}(w)$ vs Log w on a Normal Probability Scale, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$.



4. Approximating the Distribution of W by a Lognormal Distribution.

Let $LN(\eta, \tau^2)$ denote a lognormal distribution corresponding to the normal distribution $N(\eta, \tau^2)$. We consider a lognormal approximation to the distribution of W, with parameters η and σ^2 determined so that the first two moments of $LN(\eta, \tau^2)$ and of W coincide. In Figure 3 the distribution of W (in the standard case) is plotted on a normal probability paper versus log W, for $\rho = -.2(.2).8$. We see that in the standard case the distribution $F_{\underline{\theta}}(w)$ for nonnegative ρ values is very close to a lognormal distribution. The lognormal approximation is very good for $\rho = -.20$.

We consider now the lognormal approximation to $F_{\underline{\theta}}(w)$. By the methods of moment equations we determine η and τ^2 by equating the first two moments of W to those of $LN(\eta, \tau^2)$. The equations to be solved are:

$$\exp\{\eta + \tau^2/2\} = \exp\{\mu_1 + \frac{\sigma_1^2}{2}\} + \exp\{\mu_2 + \frac{\sigma_2^2}{2}\},$$

(4.1) and

$$\exp\{2\eta + 2\tau^2\} = \exp\{2\mu_1 + 2\sigma_1^2\} + \exp\{2\mu_2 + 2\sigma_2^2\}$$
$$+ 2\exp\{\mu_1 + \mu_2 + \tfrac{1}{2}(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)\}.$$

These equations yield the solutions:

$$\tau^2 = \log \frac{e^{2\mu_1+2\sigma_1^2} + e^{2\mu_2+2\sigma_2^2} + 2e^{\mu_1+\mu_2+\frac{1}{2}(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)}}{e^{2\mu_1+\sigma_1^2} + e^{2\mu_2+\sigma_2^2} + 2e^{\mu_1+\mu_2+\frac{1}{2}(\sigma_1^2+\sigma_2^2)}} \tag{4.2}$$

and

$$\eta = \log(e^{\mu_1+\sigma_1^2/2} + e^{\mu_2+\sigma_2^2/2}) - \tau^2/2. \tag{4.3}$$

The approximation to the distribution of W is then

$$F_{\underline{\theta}}(w) \doteq \phi\left(\frac{\log w - \eta}{\tau}\right). \tag{4.4}$$

In Table 7 we compare the exact distribution of W in the standard case with the lognormal approximation for $\rho = -.75(.25).75$. Table 7 confirms the earlier conclusion from Figure 3, that the lognormal approximation (4.4) is very good for nonnegative values of ρ. In Table 8 we provide a comparison between the exact distribution and the lognormal approximation in the case of $\mu_1 = \mu_2 = 1$ and $\sigma_1 = \sigma_2 = 1$. Here also the approximation is also good ρ nonnegative values of ρ. The extent to which the lognormal deviates from the exact in the case of $\rho = -.75$ is shown in Figure 4. Due to the asymmetry of the distribution, the lognormal distribution provides a better approximation at the right hand tail of the distribution than at the left hand tail. Between the 10th and 90th percentiles the lognormal distribution is good even in the case of $\rho = -.75$.

In order to improve the approximation, especially for negative values of ρ, we consider the Edgeworth expansion (see Johnson and Kotz [3; pp. 17]). A 2-term approximation formula is

$$F_{\underline{\theta}}(w) \doteq \phi\left(\frac{\log w - \eta}{\tau}\right) - \frac{\gamma_1^*(\underline{\theta})}{6}\left[\left(\frac{\log w - \eta}{\tau}\right)^2 - 1\right] \phi\left(\frac{\log w - \eta}{\tau}\right); \tag{4.5}$$

and a 4-term approximation formula is

$$(4.6)\qquad F_{\underline{\theta}}(w) = G_{\underline{\theta}}^{(2)}(w) - \frac{1}{24}(\gamma_2^*(\underline{\theta}) - 3)\cdot\left[\left(\frac{\log w - \eta}{\tau}\right)^3 - 3\cdot\left(\frac{\log w - \eta}{\tau}\right)\right] \phi\left(\frac{\log w - \eta}{\tau}\right) - \frac{1}{72}\gamma_1^*(\underline{\theta})\left[\left(\frac{\log w - \eta}{\tau}\right)^5 - 10\left(\frac{\log w - \eta}{\tau}\right)^3 + 15\left(\frac{\log w - \eta}{\tau}\right)\right]\phi\left(\frac{\log w - \eta}{\tau}\right),$$

where $\gamma_1^*(\underline{\theta})$ and $\gamma_2^*(\underline{\theta})$ are the coefficients of skewness and kurtosis of $Z = \log W$, and $\phi(u)$ is the standard normal p.d.f. $G_{\underline{\theta}}^{(2)}(w)$ is the R.H.S. of (4.5). *In order to apply these approximations we have to discuss the problem* of computing the moments of $Z = \log W$, which are required for $\gamma_1^*(\underline{\theta})$ and $\gamma_2^*(\underline{\theta})$. This problem is discussed in Section 5.

In Table 9 we provide the results of a 2-term Edgeworth expansion, for the case presented in Table 8.

A 4-term approximation is given in Table 10. The comparison of these Tables with Table 9 showing sometimes certain improvements but not substantial ones.

Other types of approximations that we attempted did not yield better results.

Table 7. The Exact Distribution of W (upper) and the Lognormal Approximation (lower); $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$.

W/ρ	-.75	-.50	-.25	0	.25	.50	.75
1.000	0.00946	0.04229	0.07849	0.11346	0.14692	0.17938	0.21151
2.000	0.26969	0.32330	0.36195	0.39414	0.42283	0.44948	0.47499
3.000	0.60336	0.59423	0.59875	0.60781	0.61889	0.63107	0.64397
4.000	0.78277	0.75881	0.74750	0.74334	0.74333	0.74594	0.75029
5.000	0.86991	0.85105	0.83682	0.82774	0.82248	0.81994	0.81936
6.000	0.91602	0.90363	0.89113	0.88122	0.87402	0.86906	0.86585
7.000	0.94279	0.93491	0.92509	0.91600	0.90850	0.90262	0.89814
8.000	0.95943	0.95437	0.94698	0.93922	0.93218	0.92617	0.92119
9.000	0.97031	0.96701	0.96153	0.95512	0.94881	0.94307	0.93803
10.000	0.97771	0.97551	0.97147	0.96625	0.96074	0.95545	0.95058
1.000	0.0801	0.0939	0.1108	0.1311	0.1548	0.1819	0.2118
2.000	0.3483	0.3651	0.3840	0.4047	0.4271	0.4507	0.4751
3.000	0.5806	0.5886	0.5977	0.6078	0.6190	0.6311	0.6440
4.000	0.7338	0.7348	0.7364	0.7386	0.7416	0.7455	0.7503
5.000	0.8292	0.8265	0.8240	0.8218	0.8202	0.8194	0.8193
6.000	0.8883	0.8842	0.8799	0.8757	0.8718	0.8685	0.8658
7.000	0.9255	0.9211	0.9163	0.9114	0.9066	0.9021	0.8981
8.000	0.9493	0.9452	0.9406	0.9356	0.9306	0.9257	0.9212
9.000	0.9649	0.9612	0.9570	0.9524	0.9476	0.9427	0.9380
10.000	0.9753	0.9721	0.9684	0.9643	0.9598	0.9552	0.9506

Table 8. The Exact Distribution of W (upper) and Its Lognormal Approximation (lower) for $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$.

W/ρ	-1.00	-.75	-.50	-.25	0	.25	.50	.75	1.00
1	0.0000	0.0000	0.0001	0.0014	0.0049	0.0110	0.0197	0.0310	0.0452
2	0.0000	0.0008	0.0106	0.0292	0.0518	0.0765	0.1025	0.1299	0.1587
3	0.4602	0.0183	0.0616	0.1034	0.1415	0.1768	0.2105	0.2435	0.2761
4	0.6141	0.0873	0.1560	0.2061	0.2473	0.2835	0.3168	0.3486	0.3795
5	0.7094	0.2080	0.2715	0.3156	0.3514	0.3829	0.4119	0.4397	0.4666
6	0.7743	0.3481	0.3876	0.4188	0.4458	0.4705	0.4939	0.5168	0.5393
7	0.8208	0.4795	0.4927	0.5102	0.5280	0.5458	0.5635	0.5816	0.5998
8	0.8551	0.5893	0.5826	0.5883	0.5981	0.6096	0.6223	0.6360	0.6504
9	0.8812	0.6758	0.6570	0.6540	0.6573	0.6636	0.6719	0.6818	0.6929
10	0.9014	0.7421	0.7176	0.7087	0.7070	0.7091	0.7138	0.7206	0.7289
11	0.9173	0.7924	0.7666	0.7540	0.7487	0.7476	0.7494	0.7536	0.7595
12	0.9300	0.8307	0.8060	0.7914	0.7837	0.7802	0.7797	0.7817	0.7857
13	0.9403	0.8600	0.8378	0.8225	0.8132	0.8079	0.8056	0.8059	0.8083
14	0.9487	0.8828	0.8635	0.8483	0.8380	0.8315	0.8279	0.8268	0.8279
15	0.9557	0.9008	0.8843	0.8698	0.8591	0.8517	0.8470	0.8449	0.8449
16	0.9614	0.9152	0.9014	0.8877	0.8770	0.8691	0.8636	0.8606	0.8598
17	0.9663	0.9268	0.9154	0.9028	0.8922	0.8840	0.8781	0.8744	0.8729
18	0.9704	0.9364	0.9269	0.9155	0.9053	0.8970	0.8906	0.8864	0.8844
19	0.9739	0.9443	0.9366	0.9262	0.9164	0.9082	0.9017	0.8971	0.8946
20	0.9768	0.9510	0.9446	0.9353	0.9261	0.9180	0.9113	0.9065	0.9036
21	0.9794	0.9566	0.9514	0.9431	0.9344	0.9265	0.9199	0.9148	0.9117
22	0.9816	0.9615	0.9572	0.9498	0.9417	0.9340	0.9274	0.9222	0.9189
23	0.9835	0.9656	0.9621	0.9555	0.9480	0.9406	0.9341	0.9288	0.9254
24	0.9852	0.9692	0.9663	0.9605	0.9534	0.9464	0.9400	0.9348	0.9312
25	0.9867	0.9723	0.9699	0.9648	0.9583	0.9516	0.9453	0.9401	0.9365
1	0.0013	0.0021	0.0033	0.0052	0.0084	0.0133	0.0207	0.0311	0.0452
2	0.0257	0.0319	0.0403	0.0512	0.0654	0.0831	0.1046	0.1299	0.1587
3	0.0913	0.1039	0.1192	0.1378	0.1596	0.1847	0.2128	0.2435	0.2761
4	0.1852	0.2008	0.2191	0.2402	0.2641	0.2904	0.3188	0.3487	0.3795
5	0.2888	0.3042	0.3219	0.3420	0.3641	0.3881	0.4135	0.4398	0.4666
6	0.3897	0.4030	0.4182	0.4352	0.4538	0.4739	0.4951	0.5170	0.5393
7	0.4816	0.4920	0.5039	0.5172	0.5318	0.5476	0.5644	0.5819	0.5998
8	0.5622	0.5696	0.5781	0.5877	0.5985	0.6102	0.6229	0.6364	0.6504
9	0.6314	0.6360	0.6414	0.6477	0.6550	0.6632	0.6724	0.6823	0.6929
10	0.6899	0.6921	0.6950	0.6985	0.7029	0.7081	0.7142	0.7212	0.7289
11	0.7390	0.7393	0.7401	0.7414	0.7434	0.7461	0.7497	0.7542	0.7595
12	0.7800	0.7789	0.7781	0.7776	0.7776	0.7784	0.7799	0.7824	0.7857
13	0.8143	0.8121	0.8100	0.8082	0.8067	0.8059	0.8058	0.8066	0.8083
14	0.8429	0.8399	0.8369	0.8340	0.8315	0.8294	0.8280	0.8275	0.8279
15	0.8667	0.8633	0.8596	0.8560	0.8526	0.8496	0.8472	0.8456	0.8449
16	0.8867	0.8829	0.8788	0.8747	0.8707	0.8670	0.8638	0.8614	0.8598
17	0.9034	0.8994	0.8951	0.8907	0.8862	0.8820	0.8782	0.8751	0.8729
18	0.9174	0.9134	0.9090	0.9043	0.8996	0.8950	0.8908	0.8872	0.8844
19	0.9292	0.9252	0.9208	0.9160	0.9111	0.9063	0.9018	0.8979	0.8946
20	0.9392	0.9352	0.9309	0.9261	0.9212	0.9162	0.9115	0.9072	0.9036
21	0.9476	0.9438	0.9395	0.9348	0.9299	0.9249	0.9200	0.9156	0.9117
22	0.9547	0.9511	0.9469	0.9424	0.9375	0.9325	0.9276	0.9230	0.9189
23	0.9608	0.9573	0.9534	0.9489	0.9441	0.9392	0.9343	0.9296	0.9254
24	0.9660	0.9627	0.9589	0.9546	0.9500	0.9451	0.9402	0.9355	0.9312
25	0.9704	0.9673	0.9637	0.9596	0.9551	0.9503	0.9455	0.9408	0.9365

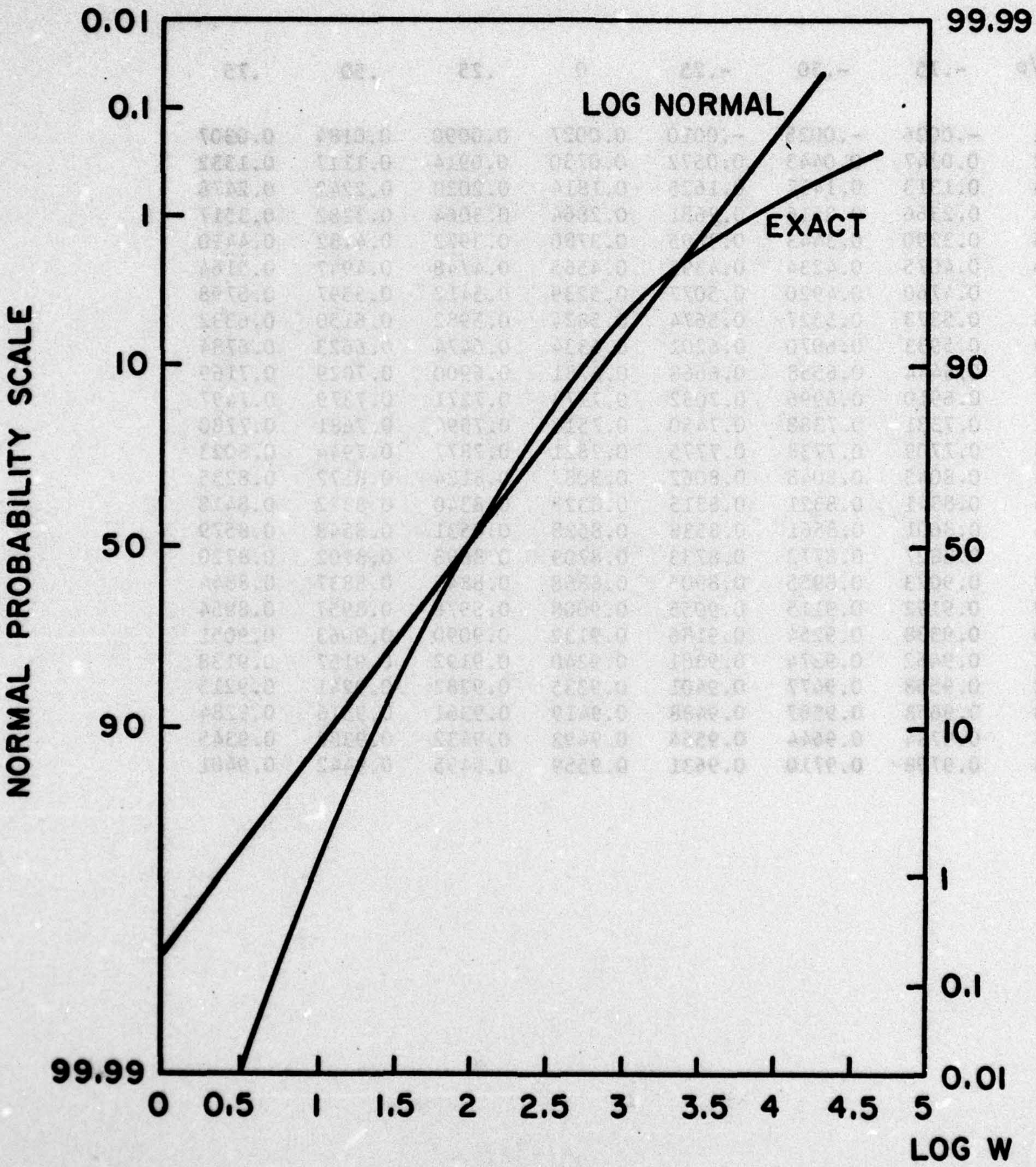
Table 9. A 2-term Edgeworth Expansion Approximation $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$.

W/ρ	-.75	-.50	-.25	0	.25	.50	.75
1	0.0042	0.0050	0.0066	0.0094	0.0139	0.0209	0.0312
2	0.0397	0.0452	0.0541	0.0669	0.0837	0.1048	0.1300
3	0.1086	0.1214	0.1384	0.1596	0.1845	0.2126	0.2434
4	0.1971	0.2158	0.2376	0.2623	0.2893	0.3183	0.3485
5	0.2927	0.3141	0.3369	0.3611	0.3865	0.4128	0.4396
6	0.3867	0.4078	0.4289	0.4503	0.4722	0.4944	0.5168
7	0.4742	0.4928	0.5106	0.5282	0.5459	0.5637	0.5817
8	0.5526	0.5676	0.5816	0.5952	0.6087	0.6224	0.6362
9	0.6212	0.6324	0.6425	0.6522	0.6619	0.6719	0.6821
10	0.6803	0.6878	0.6943	0.7006	0.7071	0.7138	0.7210
11	0.7308	0.7348	0.7383	0.7417	0.7453	0.7494	0.7541
12	0.7735	0.7747	0.7755	0.7765	0.7779	0.7797	0.7823
13	0.8095	0.8083	0.8071	0.8061	0.8056	0.8057	0.8066
14	0.8398	0.8367	0.8338	0.8313	0.8293	0.8280	0.8275
15	0.8652	0.8607	0.8565	0.8528	0.8497	0.8472	0.8456
16	0.8865	0.8809	0.8758	0.8712	0.8672	0.8639	0.8614
17	0.9044	0.8980	0.8922	0.8870	0.8823	0.8783	0.8752
18	0.9194	0.9125	0.9063	0.9006	0.8954	0.8910	0.8873
19	0.9320	0.9248	0.9183	0.9123	0.9069	0.9020	0.8979
20	0.9426	0.9353	0.9286	0.9225	0.9168	0.9117	0.9073
21	0.9515	0.9442	0.9375	0.9313	0.9255	0.9203	0.9156
22	0.9590	0.9518	0.9452	0.9390	0.9332	0.9278	0.9231
23	0.9653	0.9583	0.9518	0.9457	0.9399	0.9345	0.9297
24	0.9706	0.9638	0.9575	0.9516	0.9458	0.9405	0.9356
25	0.9751	0.9686	0.9625	0.9567	0.9511	0.9458	0.9409

Table 10. 4-term Edgeworth Expansion Approximation. For $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$.

W/ρ	-.75	-.50	-.25	0	.25	.50	.75
1	-.0026	-.0025	-.0010	0.0027	0.0090	0.0184	0.0307
2	0.0347	0.0443	0.0572	0.0730	0.0914	0.1117	0.1332
3	0.1313	0.1457	0.1625	0.1814	0.2020	0.2242	0.2478
4	0.2366	0.2516	0.2681	0.2864	0.3064	0.3282	0.3517
5	0.3290	0.3443	0.3605	0.3780	0.3972	0.4182	0.4410
6	0.4075	0.4234	0.4395	0.4565	0.4748	0.4947	0.5164
7	0.4760	0.4920	0.5077	0.5239	0.5412	0.5597	0.5798
8	0.5373	0.5527	0.5674	0.5824	0.5982	0.6150	0.6332
9	0.5933	0.6070	0.6201	0.6334	0.6474	0.6623	0.6784
10	0.6444	0.6558	0.6668	0.6781	0.6900	0.7029	0.7169
11	0.6910	0.6996	0.7082	0.7173	0.7271	0.7379	0.7497
12	0.7331	0.7388	0.7450	0.7518	0.7594	0.7681	0.7780
13	0.7709	0.7738	0.7775	0.7821	0.7877	0.7944	0.8023
14	0.8045	0.8048	0.8062	0.8087	0.8124	0.8172	0.8235
15	0.8341	0.8321	0.8315	0.8322	0.8340	0.8372	0.8418
16	0.8601	0.8561	0.8538	0.8528	0.8531	0.8548	0.8579
17	0.8827	0.8772	0.8733	0.8709	0.8698	0.8702	0.8720
18	0.9023	0.8955	0.8905	0.8868	0.8845	0.8837	0.8844
19	0.9192	0.9115	0.9055	0.9008	0.8976	0.8957	0.8954
20	0.9338	0.9254	0.9186	0.9132	0.9090	0.9063	0.9051
21	0.9462	0.9374	0.9301	0.9240	0.9192	0.9157	0.9138
22	0.9568	0.9477	0.9401	0.9335	0.9282	0.9241	0.9215
23	0.9658	0.9567	0.9488	0.9419	0.9361	0.9316	0.9284
24	0.9734	0.9644	0.9564	0.9493	0.9432	0.9382	0.9345
25	0.9798	0.9710	0.9631	0.9559	0.9495	0.9442	0.9401

Figure 4. Normal Probability Plot vs Log W or the Exact and the Log-Normal Distributions, for $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$ and $\rho = .75$.



5. The Moments of Z = log W in the Correlated Case.

Hamdan [2] developed formulae for the expectation and variance of $Z = \log W$ in the case of correlated random variables, with possibly different variances. His formula for $E_\rho\{Z\}$ in the standard bivariate case ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$) can be written in the form

$$(5.1) \qquad \mu_1^{\rho}(Z) = \sqrt{\frac{1-\rho}{\pi}} + 2 \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} e^{j^2(1-\rho)} \Phi(-j\sqrt{2(1-\rho)}.) .$$

It is easy to prove that this series is absolutely convergent, since for large values of j $\Phi(-j\sqrt{2(1-\rho)}) \simeq \frac{1}{j\sqrt{2\pi}} e^{-j^2(1-\rho)}$ (see Feller [1; pp. 166]).

One should be careful in the computation of $\mu_1^{\rho}(Z)$ according to (5.1) since $e^{j^2(1-\rho)}$ grows very fast with j and $\Phi(-j\sqrt{2(1-\rho)})$ decreases very fast. We have found that the polynomial approximation for $\Phi(z)$ given by Zelen and Severo [7] to be very effective. This approximation is given by

$$(5.2) \qquad \Phi(z) = 1 - \phi(z) \sum_{j=1}^{5} b_j (1 + pz)^{-j}, \qquad z > 0$$

where $p = .2316419$; $b_1 = .319382$; $b_2 = -.356564$; $b_3 = 1.781478$; $b_4 = -1.821256$; and $b_5 = 1.330274$. By substituting (5.2) in (5.1) and since $\Phi(-z) = 1 - \Phi(z)$, we obtain the formula

$$(5.3) \qquad \mu_1^{\rho}(Z) = \sqrt{\frac{1-\rho}{\pi}} + \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} \sum_{i=1}^{5} \frac{b_i}{(1+p(j\sqrt{2(1-\rho)})^{i}} .$$

The convergence is of $0(\frac{1}{j^2})$. Our experience has shown that between 10 and 20 terms are sufficient in most cases to obtain stable results. The error in (5.2) is smaller in magnitude than 1.5×10^{-8} for all z. We therefore consider the values obtained from (5.3) as close to the exact ones. This approximation is better than the one given by Hamdan in [2].

We could not obtain similar formulae for the higher moments of Z. Although Hamdan provides in [2] a formula for $E_\rho\{Z^2\}$ we have not been able to apply it (the series expression given by Hamdan does not converge absolutely!). We therefore provide the following numerical approximation formula for the determination of the moments of z:

$$(5.4) \qquad \mu_r{}^\rho(z) \doteq \sum_{i=1}^{m} \sum_{j=1}^{m} [\log(e^{\eta'_i} + e^{\eta'_j})]^r \cdot$$

$$[\Phi(\eta_i,\eta_j;\rho) - \Phi(\eta_{i-1},\eta_j;\rho) - \Phi(\eta_i,\eta_{j-1},\rho) + \Phi(\eta_{i-1},\eta_{j-1};\rho)],$$

where $\Phi(z_1,z_2;\rho)$ is the standard bivariate normal integral; m is the number of subintervals for each variable. We compute the moments over a grid of m x m squares. The range in each dimension is from -4.5 to +4.5 and the length of each subinterval is $\Delta = g/m$.

In Table 11 we compare the values of the first moment of Z obtained by (5.3) and by (5.4) with m = 7.

In Table 12 we present the values of the first four moments of Z computed according to (5.4) with m = 10, and also the values of the standard deviation, $\sqrt{\gamma_1}$ and γ_2 of Z.

Table 11. The Expectation of Z.

ρ	Formula (5.3)	Formula (.54)
-.75	1.0295	1.0254
-.50	.9893	.9860
-.25	.9473	.9459
0.00	.9032	.9041
0.25	.8565	.8598
0.50	.8067	.8125
0.75	.7532	.7614

Table 12. Moments, Standard Deviation and the Coefficients of Skewness and Kurtosis of Z in the Standard Case.

ρ	-.75	-.50	-.25	0	.25	.50	.75
μ_1	1.0396	0.9998	0.9583	0.9148	0.8690	0.8203	0.7679
μ_2	1.4047	1.4165	1.4297	1.4447	1.4620	1.4824	1.5074
μ_3	2.2543	2.3395	2.4126	2.4739	2.5234	2.5601	2.5818
μ_4	4.1885	4.4522	4.7300	5.0318	5.3648	5.7353	6.1522

ρ	-.75	-.50	-.25	0	.25	.50	.75
S.D.	0.5692	0.6457	0.7151	0.7796	0.8407	0.8997	0.9580
$\sqrt{\gamma_1}$	0.6533	0.3330	0.1708	0.0848	0.0411	0.0220	0.0169
γ_2	3.9928	3.4180	3.1729	3.0648	3.0161	2.9936	2.9826

Table 12 shows again the observation previously discussed that the distribution of Z , for $\rho \geq 0$, is approximately normal. Considering the values of $\sqrt{\gamma_1}$ and γ_2 for $\rho = -.75$ it seems that the distribution of $Z = \log W$, when ρ is close to -1, can be approximated by the Pearson type IV distribution (see Johnson and Kotz [3; pp. 12]). However, it is quite difficult to compute the c.d.f. of the Pearson type IV, while the c.d.f. of Z can be computed numerically very well according to (2.5) or (2.10).

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The present paper studies the properties of the distribution of sums of dependent log-normal random variables and methods to compute numerically their corresponding c.d.f.'s. The dependence between the log-normal variables is defined in terms of the correlation between the corresponding normal variables. Two methods for numerical computations of the exact cumulative distributions are developed first. One can be described as a numerical convolution and the other is a Gauss-Legendre quadrature. These

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methods are compared by numerical results in standard and non-standard cases. The moments of the distribution of the sum are given explicitly and also the coefficients of skewness and kurtosis. It is shown that for for positive correlations the distribution of the sum is approximately log-normal. For negative values of the correlation the log-normal becomes ineffective. Another approximation is given for these cases, based on the first few terms of an Edgeworth expansion. Finally, methods for computing the moments of the logarithm of the sum are developed.